

A functional limit theorem for excited random walks

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Abstract

We consider the limit behavior of an excited random walk (ERW), i.e., a random walk whose transition probabilities depend on the number of times the walk has visited to the current state. We prove that an ERW being naturally scaled converges in distribution to an excited Brownian motion that satisfies an SDE, where the drift of the unknown process depends on its local time. Similar result was obtained by Raimond and Schapira, their proof was based on the Ray-Knight type theorems. We propose a new method of investigations based on a study of the Radon-Nikodym density of the ERW distribution with respect to the distribution of a symmetric random walk.

Key words: Excited random walks; excited Brownian motion; invariance principle.

1 Introduction and results

Let $\{X(k), k \geq 0\}$ be a sequence of \mathbb{Z} -valued random variables such that $|X(k+1) - X(k)| = 1, k \geq 0$. Denote by $\mathcal{F}_n := \sigma(X(0), X(1), \dots, X(n))$ the filtration generated by $\{X(k)\}$.

Definition 1. A random walk (RW) $\{X(k)\}$ is called an excited random walk (ERW) associated with a (may be random) sequence $\{\varepsilon_i, i \geq 0\} \subset (-1, 1)$ if

$$\mathbb{P}(X(k+1) - X(k) = 1 | \mathcal{F}_k) = 1 - \mathbb{P}(X(k+1) - X(k) = -1 | \mathcal{F}_k) = p_i, \quad (1)$$

where $i = |\{j \leq k : X(j) = X(k)\}|$, $p_i = \frac{1}{2}(1 + \varepsilon_i)$.

Note that $\{X(k)\}$ is not a Markov chain, generally, and the study of traditional topics of the theory of stochastic processes such as recurrence, invariance principles, etc., is a non-trivial one for ERW. It demands new ideas and approaches, see for example [1, 4, 6, 7, 9, 11, 13] and references therein.

It was proved by Raimond and Schapira [11] that if $\varepsilon_i = \varepsilon_i^{(n)} = \frac{1}{\sqrt{n}}\varphi(\frac{i}{\sqrt{n}})$, where φ is a bounded Lipschitz function, then the sequence of processes $\{X_n(t) := \frac{X^{(n)}(\lfloor nt \rfloor)}{\sqrt{n}}, t \geq 0\}_{n \geq 1}$ converges in distribution in $D([0, \infty))$ to excited Brownian motion that is a solution to the following SDE

$$dY(t) = \varphi(L_Y(t, Y_t))dt + dW(t),$$

where W is a Wiener process, $L_Y(t, x)$ is the local time of Y at x .

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They studied the process $\nu(i, k) = |\{j \leq i : X(j) = k\}|$ as a function of the spatial coordinate k . It was proved that some scaling of ν taken at some Markov moments converges to a solution of a Bessel type SDE that appears in a spirit of the Ray-Knight theorem, see also [8]. Then a sequence $X(k)$ (and a process Y_t) were reconstructed from ν (and the local time L_Y , respectively). The corresponding proofs used the neat martingale technique. However the number of details they have checked was really large.

We propose a different method for proving of the corresponding result. We study the Radon-Nikodym density of $\{X(k), 0 \leq k \leq n\}$ with respect to the distribution of a symmetric RW. Then we use Gikhman and Skorokhod result [5] on absolute continuity of the limit process together with the Skorokhod theorem on a single probability space, and invariance principle for the local times of random walks [3].

This method was used in [10] for studying the limit behavior of an RW with modifications at 0 whose transition probabilities are defined as in (1), where

$$i = |\{j \leq k : X(j) = 0\}|, \quad p_i = \left(\frac{1}{2} + i\Delta\right) \wedge 1,$$

$\Delta > 0$ is a size of modifications. It was proved there that $X_n \Rightarrow X_\infty$ in the scheme of series, where $\Delta_n = cn^{-\alpha}$, $c > 0$, $\alpha > 0$,

$$X_n(t) = \begin{cases} \frac{X_{\Delta_n}([nt])}{\sqrt{n}}, & \alpha \geq 1, \\ \frac{X_{\Delta_n}([nt])}{n^{1-\frac{\alpha}{2}}}, & \alpha \in (0, 1), \end{cases} \quad X_\infty(t) = \begin{cases} W(t), & \alpha > 1, \\ \sqrt{c} \int_0^t L_{X_\infty}(s, 0) ds + W(t), & \alpha = 1, \\ \eta t, & \alpha \in (0, 1), \end{cases}$$

η is a non-negative random variable with the distribution function

$$P(\eta \leq x) = 1 - e^{-\frac{x^2}{2}}, \quad x \geq 0.$$

2 Main Result and Proofs

Let $\{\omega_k\}$ be a stationary ergodic sequence. Consider a sequence of ERWs $\{X^{(n)}(k), k \geq 0\}_{n \geq 1}$ such that for a fixed $\omega = \{\omega_k\}$ the quenched probability satisfies the condition

$$P_\omega \left(X^{(n)}(k+1) - X^{(n)}(k) = 1 | \mathcal{F}_k^{(n)} \right) = 1 - P_\omega \left(X^{(n)}(k+1) - X^{(n)}(k) = -1 | \mathcal{F}_k^{(n)} \right) = p_{i,k}^{(n)}, \quad (2)$$

where

$$\mathcal{F}_k^{(n)} := \sigma(X^{(n)}(0), X^{(n)}(1), \dots, X^{(n)}(k)), \quad i = |\{j \leq k : X^{(n)}(j) = X^{(n)}(k)\}|,$$

$$p_{i,k}^{(n)} = \frac{1}{2} (1 + \varepsilon_{k, X^{(n)}(k), i}^{(n)}), \quad \varepsilon_{k, x, i}^{(n)} = n^{-1/2} \varphi\left(\frac{k}{n}, \frac{x}{\sqrt{n}}, \frac{i}{\sqrt{n}}, \omega_k\right)$$

Here φ is a fixed bounded measurable function.

The annealed, or averaged, probability will be denoted by P .

Set $X_n(t) = \frac{X^{(n)}([nt])}{\sqrt{n}}$, $n \in \mathbb{N}$, $t \geq 0$. For convenience we will assume that $X_n(0) = 0$.

Let $D([0, \infty))$ be the space of cadlag functions equipped with the Skorokhod J_1 topology, see [2].

Theorem 1. Assume that the function $\varphi : [0, \infty) \times \mathbb{R} \times [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is bounded and uniformly continuous. Then the sequence $\{X_n(\cdot), n \geq 1\}$ converges in distribution in $D([0, \infty))$ with respect to almost every quenched measure P_ω , and also with respect to the averaged measure P , to a solution of the SDE

$$Y_t = \int_0^t \bar{\varphi}(s, Y_s, L_Y(s, Y_s)) ds + W(t), \quad t \geq 0, \quad (3)$$

where $\bar{\varphi}(t, x, l) = E\varphi(t, x, l, \omega_k)$, W is a Wiener process.

Remark 1. There is a unique weak solution to (3) by Girsanov's theorem, see [8].

Proof. In order to explain the idea of the proof and to avoid cumbersome calculations, at first we prove the theorem for φ that depends only on the first three of its coordinates, i.e., $\varphi(t, x, l, \omega) = \varphi(t, x, l)$. Then we explain how to handle the general case.

Denote by $\{S(k), k \geq 0\}$ a symmetric random walk, $S(k) = \xi_1 + \dots + \xi_k$, $S(0) = 0$, where $\{\xi_i\}$ are i.i.d., $P(\xi_i = \pm 1) = 1/2$.

Let $P_{X^{(n)}}$ be a distribution of $\{X^{(n)}(k)\}_{k=0}^n$, $P_{S^{(n)}}$ be a distribution of $\{S(k)\}_{k=0}^n$,

Then $P_{X^{(n)}} \ll P_{S^{(n)}}$ and the Radon-Nikodym density equals:

$$\forall i_0 = 0, i_1, \dots, i_n \in \mathbb{Z}, |i_{k+1} - i_k|, \quad \frac{dP_{X^{(n)}}}{dP_{S^{(n)}}}(i_0, i_1, \dots, i_n) = \frac{\prod_{k=0}^{n-1} \frac{1}{2}(1 + \varepsilon_k^{(n)})}{\frac{1}{2}} = \prod_{k=0}^{n-1} (1 + \varepsilon_k^{(n)}) = \quad (4)$$

$$\prod_{k=0}^{n-1} \left(1 + \frac{1}{\sqrt{n}} \varphi\left(\frac{k}{n}, \frac{i_k}{\sqrt{n}}, \frac{l(k, i_k)}{\sqrt{n}}\right) \mathbb{1}_{i_{k+1} - i_k = 1} - \frac{1}{\sqrt{n}} \varphi\left(\frac{k}{n}, \frac{i_k}{\sqrt{n}}, \frac{l(k, i_k)}{\sqrt{n}}\right) \mathbb{1}_{i_{k+1} - i_k = -1} \right) =$$

$$\prod_{k=0}^{n-1} \left(1 + \frac{1}{\sqrt{n}} \varphi\left(\frac{k}{n}, \frac{i_k}{\sqrt{n}}, \frac{l(k, i_k)}{\sqrt{n}}\right) (i_{k+1} - i_k) \right),$$

where $l(k, i) = |\{j \leq k : X^{(n)}(j) = i\}|$

Hence

$$\frac{dP_{X^{(n)}}}{dP_{S^{(n)}}}(S(0), S(1), \dots, S(n)) =$$

$$\prod_{k=0}^{n-1} \left(1 + \frac{1}{\sqrt{n}} \varphi\left(\frac{k}{n}, \frac{S(k)}{\sqrt{n}}, \frac{\nu(k, S(k))}{\sqrt{n}}\right) \xi_{k+1} \right), \quad (5)$$

where $\nu(k, i) = |\{j \leq k : S(j) = i\}|$.

Lemma 1. Let $\{X^n, n \geq 1\}$ and $\{Y^n, n \geq 1\}$ be sequences of random elements given on the same probability space and taking values in a complete separable metric space E .

Assume that

1) $Y_n \xrightarrow{P} Y_0, n \rightarrow \infty$;

2) for each $n \geq 1$ we have the absolute continuity of the distributions

$$P_{X_n} \ll P_{Y_n};$$

3) the sequence $\{\rho_n(Y_n), n \geq 1\}$ converges in probability to a random variable p , where $\rho_n = \frac{dP_{X_n}}{dP_{Y_n}}$ is the Radon-Nikodym density;

4) $E p = 1$.

Then the sequence of distributions $\{P_{X_n}\}$ converges weakly as $n \rightarrow \infty$ to the probability measure $E(p | Y_0 = y) P_{Y_0}(dy)$.

The idea of the proof of the lemma is due to Gikhman and Skorokhod [5]. Since $\{\rho_n(Y_n), n \geq 1\}$ are non-negative random variables $E\rho_n(Y_n) = 1$, the condition $Ep = 1$ yields the uniform integrability of $\{\rho_n(Y_n), n \geq 1\}$. The proof of Lemma 1 follows from the next calculations

$$\begin{aligned} \forall f \in C_b(E) : \quad \lim_{n \rightarrow \infty} \int_E f dP_{X_n} &= \lim_{n \rightarrow \infty} Ef(X_n) = \lim_{n \rightarrow \infty} Ef(Y_n)\rho_n(Y_n) = Ef(Y_0)p = \\ E(f(Y_0)E(p|Y_0)) &= \int_E f(y)E(p|Y_0=y)P_{Y_0}(dy). \end{aligned} \quad (6)$$

Let us continue the proof of Theorem 1. We will prove convergence in distribution $\frac{X^{(n)}([n \cdot])}{\sqrt{n}} \Rightarrow Y$ in $D([0, 1])$ only.

We need the following invariance principle for RWs and the local times of RWs.

Theorem 2. *There is a probability space and copies $\{S^{(n)}(k), k = 0, \dots, n\} \stackrel{d}{=} \{S(k), k = 0, \dots, n\}$ defined on this space, and a Wiener process $W(t), t \in [0, 1]$, such that*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, 1]} \left| \frac{S^{(n)}([nt])}{\sqrt{n}} - W(t) \right| = 0, \quad (7)$$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, 1]} \sup_{x \in \mathbb{R}} \left| \frac{\nu^{(n)}([nt], [x\sqrt{n}])}{\sqrt{n}} - L_W(t, x) \right| = 0, \quad (8)$$

with probability 1, where $\nu^{(n)}(k, i) = |\{j \leq k : S^{(n)}(j) = i\}|$, L_W is the local time of the Wiener process (we consider a modification of L_W that is continuous in t, x).

Let us apply Lemma 1, where

$$X_n = X_n(t) = \frac{X^{(n)}([nt])}{\sqrt{n}}, \quad Y_n = S_n(t) = \frac{S^{(n)}([nt])}{\sqrt{n}}, \quad t \in [0, 1].$$

It follows from (5) that

$$\begin{aligned} \log \frac{dP_{X_n}}{dP_{S_n}}(S_n) &= \\ \sum_{k=0}^{n-1} \log \left(1 + \frac{1}{\sqrt{n}} \varphi\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) \xi_{k+1}^{(n)} \right) &= \\ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) \xi_{k+1}^{(n)} - \frac{1}{2n} \sum_{k=0}^{n-1} \varphi^2\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) + \\ \frac{\theta}{3n^{3/2}} \sum_{k=0}^{n-1} \left| \varphi^3\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) \right| &= I_1^n + I_2^n + I_3^n, \end{aligned}$$

where $\theta \in (-1, 1)$. Since φ is bounded, $\lim_{n \rightarrow \infty} I_3^n = 0$ for all ω .

By (7), (8), continuity of $L_W(t, x)$ in both of its arguments, and dominated convergence theorem we have convergence

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{k=0}^{n-1} \varphi^2\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) = \quad (9)$$

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \int_0^1 \varphi^2\left(\frac{[nt]}{n}, \frac{S^{(n)}(\frac{[nt]}{n})}{\sqrt{n}}, \frac{\nu^{(n)}([nt], S^{(n)}([nt]))}{\sqrt{n}}\right) dt =$$

$$\frac{1}{2} \int_0^1 \varphi^2(t, W(t), L_W(t, W(t))) dt.$$

Lemma 2. *We have convergence in probability*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) \xi_{k+1}^{(n)} \xrightarrow{P} \int_0^1 \varphi(t, W(t), L_W(t, W(t))) dW(t), \quad n \rightarrow \infty.$$

Proof. We use idea of Skorokhod [12, Chapter 3, §3]. Let $m \in \mathbb{N}$ be fixed. Then

$$\left| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) \xi_{k+1}^{(n)} - \int_0^1 \varphi(t, W(t), L_W(t, W(t))) dW(t) \right| \leq$$

$$\left| \sum_{j=0}^{m-1} \sum_{\lfloor \frac{jn}{m} \rfloor \leq k < \lfloor \frac{(j+1)n}{m} \rfloor} \left(\varphi\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) - \right. \right.$$

$$\left. \varphi\left(\frac{[jn/m]}{n}, \frac{S^{(n)}(\lfloor \frac{jn}{m} \rfloor)}{\sqrt{n}}, \frac{\nu^{(n)}(\lfloor \frac{jn}{m} \rfloor, S^{(n)}(\lfloor \frac{jn}{m} \rfloor))}{\sqrt{n}}\right) \right) \frac{\xi_{k+1}^{(n)}}{\sqrt{n}} \Bigg| +$$

$$\left| \sum_{j=0}^{m-1} \left(\varphi\left(\frac{[jn/m]}{n}, \frac{S^{(n)}(\lfloor \frac{jn}{m} \rfloor)}{\sqrt{n}}, \frac{\nu^{(n)}(\lfloor \frac{jn}{m} \rfloor, S^{(n)}(\lfloor \frac{jn}{m} \rfloor))}{\sqrt{n}}\right) \right. \right.$$

$$\left. \left(\left(\sum_{\lfloor \frac{jn}{m} \rfloor \leq k < \lfloor \frac{(j+1)n}{m} \rfloor} \frac{\xi_{k+1}^{(n)}}{\sqrt{n}} \right) - \left(W\left(\frac{[(j+1)n/m]}{n}\right) - W\left(\frac{[jn/m]}{n}\right) \right) \right) \right| +$$

$$\left| \sum_{j=0}^{m-1} \left(\varphi\left(\frac{[jn/m]}{n}, \frac{S^{(n)}(\lfloor \frac{jn}{m} \rfloor)}{\sqrt{n}}, \frac{\nu^{(n)}(\lfloor \frac{jn}{m} \rfloor, S^{(n)}(\lfloor \frac{jn}{m} \rfloor))}{\sqrt{n}}\right) - \right. \right.$$

$$\left. \varphi\left(\frac{[jn/m]}{n}, W\left(\frac{[jn/m]}{n}\right), L_W\left(\frac{[jn/m]}{n}, W\left(\frac{[jn/m]}{n}\right)\right) \right) \right.$$

$$\left. \left(W\left(\frac{[(j+1)n/m]}{n}\right) - W\left(\frac{[jn/m]}{n}\right) \right) \right| +$$

$$\left| \sum_{j=0}^{m-1} \int_{\frac{[jn/m]}{n}}^{\frac{[(j+1)n/m]}{n}} \left(\varphi\left(\frac{[jn/m]}{n}, W\left(\frac{[jn/m]}{n}\right), L_W\left(\frac{[jn/m]}{n}, W\left(\frac{[jn/m]}{n}\right)\right) - \right. \right.$$

$$\left. \left. \varphi(t, W(t), L_W(t, W(t))) \right) dW(t) \right| =$$

$$= I_1^{n,m} + I_2^{n,m} + I_3^{n,m} + I_4^{n,m}.$$

It follows from Theorem 2, Lebesgue dominated convergence theorem, and continuity of $L_W(t, x)$ in both of its arguments that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}(I_1^{n,m})^2 &= \\ \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \sum_{j=0}^{m-1} \sum_{\lfloor \frac{jn}{m} \rfloor \leq k < \lfloor \frac{(j+1)n}{m} \rfloor} \left(\varphi\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) \right. \\ &\quad \left. - \varphi\left(\frac{\lfloor jn/m \rfloor}{n}, \frac{S^{(n)}(\lfloor \frac{jn}{m} \rfloor)}{\sqrt{n}}, \frac{\nu^{(n)}(\lfloor \frac{jn}{m} \rfloor, S^{(n)}(\lfloor \frac{jn}{m} \rfloor))}{\sqrt{n}}\right) \right)^2 = \\ \mathbb{E} \sum_{j=0}^{m-1} \int_{\frac{j}{m}}^{\frac{(j+1)}{m}} \left(\varphi(t, W(t), L_W(t, W(t))) - \varphi\left(\frac{j}{m}, W\left(\frac{j}{m}\right), L_W\left(\frac{j}{m}, W\left(\frac{j}{m}\right)\right)\right) \right)^2 dt. \\ &= \lim_{n \rightarrow \infty} \mathbb{E}(I_4^{m,n})^2 \end{aligned} \quad (10)$$

It follows from Theorem 2 that $\lim_{n \rightarrow \infty} I_2^{n,m} = \lim_{n \rightarrow \infty} I_3^{n,m} = 0$ a.s. for each fixed m . So, by dominated convergence theorem

$$\forall m \geq 1 \quad \lim_{n \rightarrow \infty} \mathbb{E}(I_2^{n,m})^2 = \lim_{n \rightarrow \infty} \mathbb{E}(I_3^{n,m})^2 = 0.$$

So for any $m \geq 1$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{E} \left(\sum_{k=0}^{n-1} \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}\right) \xi_{k+1}^{(n)} - \int_0^1 \varphi(t, W(t), L_W(t, W(t))) dW(t) \right)^2 \leq \\ 4\mathbb{E} \sum_{j=0}^{m-1} \int_{\frac{j}{m}}^{\frac{(j+1)}{m}} \left(\varphi(t, W(t), L_W(t, W(t))) - \varphi\left(\frac{j}{m}, W\left(\frac{j}{m}\right), L_W\left(\frac{j}{m}, W\left(\frac{j}{m}\right)\right)\right) \right)^2 dt. \quad (11) \end{aligned}$$

Letting $m \rightarrow \infty$ we complete the proof of the lemma. \square

Since φ is bounded,

$$\mathbb{E} \exp\left\{ \int_0^1 \varphi(t, W(t), L_W(t, W(t))) dW(t) - \frac{1}{2} \int_0^1 \varphi^2(t, W(t), L_W(t, W(t))) dt \right\} = 1 \quad (12)$$

by Novikov's theorem.

Therefore, by Lemma 1 we have convergence $X_n \Rightarrow Y$, where the distribution of Y has a density $\exp\{\int_0^1 \varphi(t, W(t), L_W(t, W(t))) dW(t) - \frac{1}{2} \int_0^1 \varphi^2(t, W(t), L_W(t, W(t))) dt\}$ with respect to the Wiener measure. Note that the local time and the integrals are measurable functions with respect to the σ -algebra generated by W . So there was no necessity for calculations of the conditional expectation in Lemma 1. By Girsanov's theorem, the process Y is a weak solution to the equation (3). The theorem is proved if $\varphi(t, x, l, \omega) = \varphi(t, x, l)$.

Consider the general case.

We prove the theorem if we find the corresponding limits in (9), (10), and (11), where the general summand is replaced by

$$\varphi\left(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}, \omega_k\right),$$

and the sequence $\{\omega_k, k \geq 0\}$ is independent of $\{S^{(n)}(k)\}$.

The next statement completes the proof of the theorem.

Lemma 3. Let $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ be a uniformly continuous and bounded function, $\{\eta_k, k \geq 0\}_{n \geq 1}$ be a stationary ergodic sequence, $\{\xi_n(t), t \geq 0\}_{n \geq 1}$ be a sequence of continuous \mathbb{R}^d -valued processes that locally uniformly converge to a process $\xi(t), t \geq 0$, almost surely,

$$\forall T > 0 \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\xi_n(t) - \xi(t)| = 0 \quad a.s.$$

Then we have the following almost sure convergence

$$\forall T > 0 \quad \frac{1}{n} \sum_{k \leq nT} f(\xi_n(\frac{k}{n}), \eta_k) \xrightarrow{a.s.} \int_0^T \bar{f}(\xi(t)) dt, \quad n \rightarrow \infty,$$

where $\bar{f}(x) = \mathbb{E}f(x, \eta_k)$.

Proof. For simplicity let us prove the lemma for $T = 1$ only.

Let $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that

$$\forall x, y \in \mathbb{R}^d, |x - y| < \delta, \quad \forall z \in \mathbb{R} \quad |f(x, z) - f(y, z)| < \varepsilon.$$

Let $M > 0, N \in \mathbb{N}$ be such that

$$\mathbb{P}(\forall n \geq N \quad \sup_{t \in [0, 1]} |\xi_n(t) - \xi(t)| < \delta, \quad \sup_{t \in [0, 1]} |\xi_n(t)| \leq M) > 1 - \varepsilon.$$

Set $\Omega_\varepsilon := \{\forall n \geq N \quad \sup_{t \in [0, 1]} |\xi_n(t) - \xi(t)| < \delta, \quad \sup_{t \in [0, 1]} |\xi_n(t)| \leq M\}$.

Then for each $\omega \in \Omega_\varepsilon$ and any $m > 1/\delta$

$$\left| \frac{1}{n} \sum_{k=1}^n f(\xi_n(k/n), \eta_k) - \frac{1}{n} \sum_{j=0}^{m-1} \sum_{j/m \leq k/n < (j+1)/m} f(\xi_n(j/m), \eta_k) \right| < \varepsilon.$$

Observe that for each $\omega \in \Omega_\varepsilon$ and any $m > 1/\delta$

$$\begin{aligned} & \left| \frac{1}{n} \sum_{j=0}^{m-1} \sum_{j/m \leq k/n < (j+1)/m} (f(\xi_n(j/m), \eta_k) - \bar{f}(\xi_n(j/m))) \right| \leq \\ & \frac{2}{n} \sum_{j=0}^{m-1} \max_{p \leq M\delta} \left| \sum_{j/m \leq k/n < (j+1)/m} (f([p\delta], \eta_k) - \bar{f}([p\delta])) \right| + 2\varepsilon. \end{aligned}$$

Since f is bounded, by the ergodic theorem, for any fixed m we have the convergence

$$\frac{2}{n} \sum_{j=0}^{m-1} \max_{p \leq M\delta} \left| \sum_{j/m \leq k/n < (j+1)/m} (f([p\delta], \eta_k) - \bar{f}([p\delta])) \right| \xrightarrow{a.s.} 0, \quad n \rightarrow \infty.$$

It follows from the previous estimates that for a.a. $\omega \in \Omega_\varepsilon$ and all $m > 1/\delta$

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{k=1}^n f(\xi_n(k/n), \eta_k) - \int_0^1 \bar{f}(\xi(t)) dt \right| \leq \\ & \limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{k=1}^n f(\xi_n(k/n), \eta_k) - \frac{1}{n} \sum_{j=0}^{m-1} \sum_{j/m \leq k/n < (j+1)/m} f(\xi_n(j/m), \eta_k) \right| + \end{aligned}$$

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{j=0}^{m-1} \sum_{j/m \leq k/n < (j+1)/m} f(\xi_n(j/m), \eta_k) - \frac{1}{n} \sum_{j=0}^{m-1} \sum_{j/m \leq k/n < (j+1)/m} \bar{f}(\xi_n(j/m)) \right| + \\
& \limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{j=0}^{m-1} \sum_{j/m \leq k/n < (j+1)/m} \bar{f}(\xi_n(j/m)) - \frac{1}{n} \sum_{j=0}^{m-1} \sum_{j/m \leq k/n < (j+1)/m} \bar{f}(\xi(j/m)) \right| + \\
& \limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{j=0}^{m-1} \sum_{j/m \leq k/n < (j+1)/m} \bar{f}(\xi(j/m)) - \int_0^T \bar{f}(\xi(t)) dt \right| \leq \\
& 4\varepsilon + \left| \frac{1}{m} \sum_{j=0}^{m-1} \bar{f}(\xi(j/m)) - \int_0^1 \bar{f}(\xi(t)) dt \right|.
\end{aligned}$$

Passing $m \rightarrow \infty$ we get for a.a. $\omega \in \Omega_\varepsilon$

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{k=1}^n f(\xi_n(k/n), \eta_k) - \int_0^1 \bar{f}(\xi(t)) dt \right| \leq 4\varepsilon.$$

Since $\varepsilon > 0$ were arbitrary, this completes the proof of Lemma 3 and hence Theorem 1. \square

\square

Remark 2. Assumption of boundedness and uniform continuity of φ may be relaxed.

We used boundedness of φ when we apply dominated convergence theorem in Lemma 2, and also when we applied Novikov's theorem to (12), or applying ergodic theorem in Lemma 3.

Using truncation arguments it can be proved that assumption of boundedness of φ can be replaced by the linear growth condition with respect to the second argument. To guarantee that $p_{i,k}^{(n)}$ in (2) is a probability we have to define it by $p_{i,k}^{(n)} = (\frac{1}{2}(1 + \varepsilon_{k, X^{(n)}(k), i, \omega_k}^{(n)})) \vee 0) \wedge 1$.

If φ depends only on the first three of its coordinates, i.e., $\varphi(t, x, l, \omega) = \varphi(t, x, l)$, we used only the continuity of φ , so the uniform continuity condition is an extra assumption.

If ω_k are bounded random variables, the uniform continuity condition can be replaced by only continuity assumption too.

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